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A New Bicubic Interpolation over Right Triangles*

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1. INTRODUCTION

C. Hall [3] has given several bicubic schemes for interpolation of a function of two variables over a right triangular domain. The formulae may be used in conjunction with the bicubic Hermite interpolation on a rectangle [1, 3] for purposes of piecewise interpolation over certain polygonal domains. Similarly they may be used to construct approximating subspaces for use in applications of the *Ritz method* (or finite element method) to the variational solution of boundary value problems following Birkhoff, Schultz and Varga [1].

The purpose of this paper is to present a modification of Hall's interpolation Scheme B [3] which is useful in the variational solution of *biharmonic problems* for certain polygonal domains when a normal derivative boundary condition $\partial u/\partial n = 0$ is given. In particular, we have in mind a polygonal domain P whose boundary lies along the edges and diagonals of the elements of some rectangular mesh. Such a polygon is partitioned by this mesh into a union of rectangles and right triangles, where the hypotenuse of each triangular element coincides with an oblique segment of ∂P. An excellent survey of interpolation and approximation in such polygons is given by Birkhoff [2].

We shall be concerned with finding a bicubic interpolating polynomial $w(x, y)$ for a function $f(x, y)$ on a right triangular domain, the polynomial having the two properties

(i) $f \equiv 0$ on the hypotenuse implies $w \equiv 0$ there, and
(ii) $\partial f/\partial n \equiv 0$ on the hypotenuse implies $\partial w/\partial n \equiv 0$ there.

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These two properties of w will make it quite convenient to satisfy the "essential" boundary conditions of biharmonic problems, $w \equiv 0$ and/or $\partial w/\partial n \equiv 0$, in applications of the Ritz method. Hall's Scheme B (Section 2) has property (i), but not property (ii). The new method presented in Section 3 has both properties (cf. Theorem 2) and yields a fourth-order approximation to $f \in C^4$ (cf. Theorem 4).

2. Bicubic Hermite Interpolation Over a Right Triangle

The following interpolation scheme is due to Hall [3, Scheme B]. Let $f(x, y)$ be a function of class $C^{1,1}$ on the right triangle $T_0 : 0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b - (b/a)x$ of Fig. 1. Also let $x_0 = 0$, $x_1 = a$, $y_0 = 0$, $y_1 = b$ and

$$f_{i,j}^{(k,m)} \equiv \frac{\partial^{k+m} f}{\partial x^k \, \partial y^m} (x_i , y_j).$$

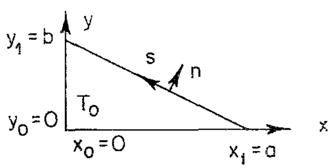

Fig 1. The Right Triangle T_0.

Then the *bicubic Hermite interpolation* to $f(x, y)$ is

$$v(x, y) = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{m=0}^{1} f_{i,j}^{(k,m)} C_{k,m;i,j}(x, y),$$
$$(0 \leqslant i + j \leqslant 1) \; 0 \leqslant x \leqslant a, \qquad 0 \leqslant y \leqslant b - \frac{b}{a} x. \tag{1}$$

The functions $C_{k,m;i,j}(x, y)$ may be found in [5, p. 9] or [3, p. 4].

From [3, Theorem 2] we know that $v(x, y)$ satisfies the 12 interpolation conditions

$$v_{i,j}^{(k,m)} = f_{i,j}^{(k,m)}, \qquad 0 \leqslant k, m \leqslant 1, \qquad 0 \leqslant i + j \leqslant 1, \tag{2}$$

and exactly matches any cubic polynomial along the hypotenuse. In fact, one can verify that, along the hypotenuse of T_0, v is the cubic Hermite interpolate of f. Also the restrictions of v and $\partial v/\partial n$ to the legs of T_0 are the cubic Hermite interpolates of f and $\partial f/\partial n$, respectively, thus insuring $C^{1,1}$ continuity across the interfaces with adjacent rectangles of P.

Hence, we have

THEOREM 1. *If* $f(x, b - (b/a)x) = 0$, $0 \leqslant x \leqslant a$, *then the same is true of* v.

3. Bicubic Interpolation with a Normal Derivative Condition

Although, along the hypotenuse of T_0, v depends only upon its values and its first derivatives' values at the end points, $\partial v/\partial n$ on the hypotenuse depends upon all 12 parameters in (2). Therefore, v has property (i) but not property (ii) of Section 1. In fact, Hall has shown [3, Corollary 1 to Theorem 1] that there is no interpolating polynomial which satisfies (2), reduces to the cubic Hermite interpolate of f and $\partial f/\partial n$ on the legs of T_0, and has the property that *both* v and $\partial v/\partial n$ along the hypotenuse are independent of $f_{0,0}^{(k,m)}$.[1] For purposes of application to biharmonic boundary value problems, the latter property is essential to the matching of boundary conditions $v \equiv 0$ and/or $\partial v/\partial n \equiv 0$ on the hypotenuse. It is also essential that $v(x, y)$ on T_0 be "compatible" with bicubic Hermite interpolation on rectangles adjacent to T_0 in P so $v \in C^{1,1}[P]$. Therefore, we are forced by Hall's result and the preceding remarks to relax the 12 interpolation conditions (2). Indeed, if we replace the 3 parameters $f_{0,0}^{(k,m)}$, $0 \leqslant k + m \leqslant 1$, by linear combinations of the remaining parameters, then it is possible to produce a *9-parameter* formula for a bicubic $w(x, y)$ such that (a) the remaining 9 values $f_{0,0}^{(1,1)}$ and $f_{i,j}^{(k,m)}$, $0 \leqslant k, m \leqslant 1$, $i + j = 1$, are interpolated, (b) w and $\partial w/\partial n$ are the cubic Hermite interpolates of these new data on the legs of T_0, and (c) w has properties (i) and (ii) of Section 1. From (c) it will follow that w and $\partial w/\partial n$ along the hypotenuse are independent of $f_{0,0}^{(k,m)}$.

Naturally, w no longer exactly matches $f_{0,0}^{(k,m)}$, $0 \leqslant k + m \leqslant 1$, but the approximations are good (cf. Theorem 4). Also it is clear that the interpolation of f on rectangles adjoining T_0 in P must match the same linear combinations which replace f, $\partial f/\partial x$ and $\partial f/\partial y$ at the common node in order to preserve $w \in C^{1,1}[P]$.

Let us define

$$w(x, y) = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{m=0}^{1} w_{i,j}^{(k,m)} C_{k,m;i,j}(x, y), \qquad (0 \leqslant i + j \leqslant 1) \quad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b - \frac{b}{a} x, \tag{3}$$

[1] Hall's Corollary actually states that there is no such *bicubic* interpolating polynomial, but it is clear that his proof holds for higher degree polynomials as well.

where the $C_{k,m;i,j}(x, y)$ are the same as in (1), the *nine independent parameters* are

$$w_{i,j}^{(k,m)} = f_{i,j}^{(k,m)}, \qquad 0 \leqslant k, m \leqslant 1, \qquad i + j = 1, \tag{4}$$

$$w_{0,0}^{(1,1)} = f_{0,0}^{(1,1)}, \tag{5}$$

and the *three dependent parameters* are taken to be

$$\begin{aligned} w_{0,0} = {} & (a^2 + b^2)^{-1} [a^2 f_{0,1} + b^2 f_{1,0}] + ab(a^2 + b^2)^{-1} \int_{(0,b)}^{(a,0)} \frac{\partial f}{\partial n}\, ds \\ & + \frac{ab}{6} [f_{0,0}^{(1,1)} + f_{0,1}^{(1,1)} + f_{1,0}^{(1,1)}], \end{aligned} \tag{6}$$

$$w_{0,0}^{(1,0)} = f_{0,1}^{(1,0)} - \frac{b}{2} [f_{0,0}^{(1,1)} + f_{0,1}^{(1,1)}], \tag{7}$$

$$w_{0,0}^{(0,1)} = f_{1,0}^{(0,1)} - \frac{a}{2} [f_{0,0}^{(1,1)} + f_{1,0}^{(1,1)}]. \tag{8}$$

Notice that $f_{0,0}^{(k,m)}$, $0 \leqslant k + m \leqslant 1$, do not appear in (4)–(8). The values of $w_{0,0}^{(k,m)}$, $0 \leqslant k + m \leqslant 1$, in (6)–(8) are constrained to be certain linear combinations of the nine independent parameters in (4)–(5) and the integral of $\partial f/\partial n$ along the hypotenuse. We shall call $w(x, y)$ the *constrained bicubic interpolation* to $f(x, y)$. From (3)–(4) v and w are identical on the hypotenuse, and Theorem 1 holds with v replaced by w.

A direct, but tedious, evaluation of (3) using (4)–(8) reveals that

$$\begin{aligned} (a^2 + b^2)^{1/2} \frac{\partial w}{\partial n}\left(x, b - \frac{b}{a}x\right) = {} & [bf_{0,1}^{(1,0)} + af_{0,1}^{(0,1)}]\left[3\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right) + 1\right] \\ & + [bf_{1,0}^{(1,0)} + af_{1,0}^{(0,1)}]\left[3\left(\frac{x}{a}\right)^2 - 2\left(\frac{x}{a}\right)\right] \\ & + \left[\int_{(0,b)}^{(a,0)} \frac{\partial f}{\partial n}\, ds\right]\left[6\left(\frac{x}{a}\right)^2 - 6\left(\frac{x}{a}\right)\right], \\ & \qquad 0 \leqslant x \leqslant a. \end{aligned} \tag{9}$$

If $\partial f/\partial n$ vanishes along the hypotenuse of T_0, then $f^{(1,0)} = (-a/b) f^{(0,1)}$ on the hypotenuse, and we have proved

THEOREM 2. *If $\partial f/\partial n(x, b - (b/a)x) = 0$, $0 \leqslant x \leqslant a$, then the same is true of $\partial w/\partial n$. Also if $f(x, b - (b/a)x) = 0$, $0 \leqslant x \leqslant a$, then $w(x, b - (b/a)x) = 0$, $0 \leqslant x \leqslant a$.*

Notice that for general *nonhomogeneous boundary values*, the cubic Hermite interpolate $w(x, b - (b/a)x)$ to $f(x, b - (b/a)x)$ is only a fourth-order approximation which is exact when $f(x, b - (b/a)x)$ is a polynomial of degree three or less. Similarly, Eq. (9) may be viewed as an interpolation formula. It presents a quadratic polynomial $\partial w/\partial n$ which is uniquely determined by $(\partial w/\partial n)_{i,j} = (\partial f/\partial n)_{i,j}$, $i+j=1$, and $\int_{(0,b)}^{(a,0)} \partial w/\partial n \, ds = \int_{(0,b)}^{(a,0)} \partial f/\partial n \, ds$. The latter condition implies that there exists a third point (ξ, η) on the hypotenuse where $\partial w/\partial n$ and $\partial f/\partial n$ agree. Therefore, from Lagrangian interpolation theory we know that $\partial w/\partial n$ is a third-order approximation to $\partial f/\partial n$ on the hypotenuse and is exact when $\partial f/\partial n(x, b - (b/a)x)$ is a polynomial of degree two or less.

4. Interpolation Error Bounds

From [3, Theorem 5] we already have

THEOREM 3 (Hall). *If $f(x, y) \in C^4[T_0]$, then the bicubic Hermite interpolation, $v(x, y)$, to $f(x, y)$ on T_0 satisfies*

$$\| v^{(k,m)} - f^{(k,m)} \|_{L^\infty[T_0]} \leqslant K_{k,m} \max_{j+l=4} \| f^{(j,l)} \|_{L^\infty[T_0]} h^4/a^k b^m, \tag{10}$$

where $0 \leqslant k+m \leqslant 3$, $h = \max\{a, b\}$ and the $K_{k,m}$ are constants.

The constrained bicubic interpolate $w(x, y)$ differs from $v(x, y)$ only in that w satisfies (6)–(8) while v satisfies $v_{0,0}^{(k,m)} = f_{0,0}^{(k,m)}$, $0 \leqslant k+m \leqslant 1$. We show next that the $w_{0,0}^{(k,m)}$ of (6)–(8) approximate the $f_{0,0}^{(k,m)}$, $0 \leqslant k+m \leqslant 1$, so closely that $w(x, y)$ is also a fourth-order approximation to $f(x, y)$.

THEOREM 4. *If $f(x, y) \in C^4[T_0]$, then the constrained bicubic interpolation, $w(x, y)$, to $f(x, y)$ on T_0 satisfies*

$$w_{0,0}^{(p,q)} = f_{0,0}^{(p,q)} + O(h^{4-p-q}), \qquad 0 \leqslant p+q \leqslant 1, \tag{11}$$

$$\| w^{(k,m)} - f^{(k,m)} \|_{L^\infty[T_0]} \leqslant M_{k,m} h^4/a^k b^m, \qquad 0 \leqslant k+m \leqslant 3, \tag{12}$$

where $h = \max\{a, b\}$ and the $M_{k,m}$ are constants.

Proof. An inspection shows that, for $(x, y) \in T_0$, $C_{k,m;i,j}(x, y) = O(a^k b^m) = O(h^{k+m})$, $0 \leqslant k$, $m \leqslant 1$, $0 \leqslant i+j \leqslant 1$. Therefore, since w and v differ only in the terms containing $w_{0,0}^{(p,q)}$ and $v_{0,0}^{(p,q)}$, $0 \leqslant p+q \leqslant 1$, it is easy to see that

$$w^{(k,m)}(x, y) - v^{(k,m)}(x, y) = \sum_{0 \leqslant p+q \leqslant 1} [w_{0,0}^{(p,q)} - v_{0,0}^{(p,q)}] \cdot (Oh^{p+q-k-m}),$$
$$0 \leqslant k+m \leqslant 3, \qquad (x, y) \in T_0. \tag{13}$$

In order to estimate $w_{0,0}^{(p,q)} - v_{0,0}^{(p,q)} = w_{0,0}^{(p,q)} - f_{0,0}^{(p,q)}$ according to (11), we first notice that the integral of $\partial^2 f/\partial x \partial y$ over T_0 is

$$
\begin{aligned}
&\int_0^a \int_0^{b-(b/a)x} f^{(1,1)}(x, y)\, dy\, dx \\
&\quad = \int_0^a \left[f^{(1,0)}\left(x, b - \frac{b}{a} x\right) - f^{(1,0)}(x, 0) \right] dx \\
&\quad = \int_{(0,b)}^{(a,0)} \frac{dx}{ds} \left[\frac{-a}{(a^2+b^2)^{1/2}} \frac{\partial f}{\partial s} + \frac{b}{(a^2+b^2)^{1/2}} \frac{\partial f}{\partial n} \right] ds \\
&\quad - \int_0^a f^{(1,0)}(x, 0)\, dx
\end{aligned}
$$

where $dx/ds = -a/(a^2 + b^2)^{1/2}$. Therefore, we have

$$
\begin{aligned}
\int_0^a \int_0^{b-(b/a)x} f^{(1,1)}(x, y)\, dy\, dx = f_{0,0} - (a^2+b^2)^{-1} [a^2 f_{0,1} + b^2 f_{1,0}] \\
- ab(a^2+b^2)^{-1} \int_{(0,b)}^{(a,0)} \frac{\partial f}{\partial n} ds. \qquad (14)
\end{aligned}
$$

Moreover, using Taylor's series expansions one can prove the following "truncated triangular prism rule"

$$
\int_0^a \int_0^{b-(b/a)x} f^{(1,1)}(x, y)\, dy\, dx = \frac{ab}{6} [f_{0,0}^{(1,1)} + f_{0,1}^{(1,1)} + f_{1,0}^{(1,1)}] + O(h^4). \qquad (15)
$$

Similarly, we may use the trapezoidal rule to show that

$$
\begin{aligned}
f_{0,1}^{(1,0)} - f_{0,0}^{(1,0)} &= \int_0^b f^{(1,1)}(0, y)\, dy \\
&= \frac{b}{2} [f_{0,0}^{(1,1)} + f_{0,1}^{(1,1)}] + O(b^3)
\end{aligned} \qquad (16)
$$

and

$$
\begin{aligned}
f_{1,0}^{(0,1)} - f_{0,0}^{(0,1)} &= \int_0^a f^{(1,1)}(x, 0)\, dx \\
&= \frac{a}{2} [f_{0,0}^{(1,1)} + f_{1,0}^{(1,1)}] + O(a^3).
\end{aligned} \qquad (17)
$$

Then (6), (14) and (15) imply that $f_{0,0} = w_{0,0} + O(h^4)$, (7) and (16) imply that $w_{0,0}^{(1,0)} = f_{0,0}^{(1,0)} + O(b^3)$, and (8) and (17) imply that $w_{0,0}^{(0,1)} = f_{0,0}^{(0,1)} + O(a^3)$. This establishes (11).

Now we use (11) in (13), recalling that $v_{0,0}^{(p,q)} = f_{0,0}^{(p,q)}$, to obtain

$$\| w^{(k,m)} - v^{(k,m)} \|_{L^\infty[T_0]} = O(h^{4-k-m}), \qquad 0 \leqslant k+m \leqslant 3. \tag{18}$$

It only remains to apply the triangle inequality to (10) and (18) in order to establish (12). Q.E.D.

5. Application in a Finite Element Method

We shall consider the numerical solution of the problem of the bending of a thin elastic equilateral triangular plate (Fig. 2) which is uniformly loaded and simply supported. According to classical Kirchhoff plate theory [6], the deflection $u(x, y)$ satisfies the boundary value problem $\nabla^4 u(x, y) = 1$, $(x, y) \in T$, $u = \partial^2 u/\partial n^2 = 0$, $(x, y) \in \partial T$. The exact solution is [6, p. 313]

$$u(x, y) = \frac{1}{96} x \left[\left(x - \frac{3}{2}\right)^2 - 3y^2\right]\left[1 - \left(x - \frac{1}{2}\right)^2 - y^2\right].$$

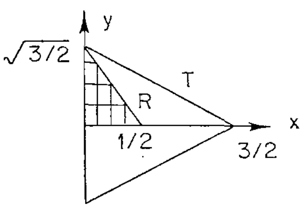

Fig. 2 Triangular Plate T

Due to the sixfold symmetry of the problem one may consider only the right triangular subregion R. Then given a rectangular mesh π on R, one may define a subspace $H_0^{(2)}(\pi)$ of piecewise bicubic polynomial functions using the usual bicubic Hermite interpolation formula for the rectangular elements and the constrained bicubic interpolation of (3)–(8) for the right triangular elements of R, such that the functions in $H_0^{(2)}(\pi)$ satisfy the essential boundary conditions $u(0, y) = \partial u/\partial n(x, 0) = \partial u/\partial n(x, -\sqrt{3}(x - \frac{1}{2})) = 0$. The algebraic and computational details of minimizing the potential energy functional $J[w] = \iint_R [\frac{1}{2}(\nabla^2 w)^2 - w]\,dxdy$ over all $w \in H_0^{(2)}(\pi)$ may be found in [4, Chap. 4].

Let the error in the approximate solution $\hat{w}(x, y)$ be

$$e(x, y) = u(x, y) - \hat{w}(x, y).$$

In Table I we summarize the results of solution using eight different uniform meshes of size h with N points on each leg of R. Table I presents the *discrete* *error norm* $\| e \|' = \max_{1 \leqslant i,j \leqslant N} | e(x_i, y_j)|$ and the *computed order of error* $\alpha = \log(\| e(h_1)\|' / \| e(h_2)\|')/\log(h_1/h_2)$ based on two successive meshes of size h_1 and h_2, i.e., $\| e \|' \approx Mh^{\alpha}$.

TABLE I

Bicubic Hermite Approximation Results

N	dim $H_0^{(2)}(\pi)$	h	$\| e \|'$	α
2	1	0.866	$0.208 \cdot 10^{-2}$	—
3	5	0.433	$0.119 \cdot 10^{-3}$	4.13
4	13	0.289	$0.225 \cdot 10^{-4}$	4.11
5	25	0.216	$0.681 \cdot 10^{-5}$	4.15
6	41	0.173	$0.271 \cdot 10^{-5}$	4.12
7	61	0.144	$0.128 \cdot 10^{-5}$	4.10
8	85	0.124	$0.684 \cdot 10^{-6}$	4.08
9	113	0.108	$0.396 \cdot 10^{-6}$	4.09

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